Math 246B Lecture 24 Notes

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1 Picard's Great Theorem and Fatou's Theorem

1.1 Picard's Great Theorem

Theorem 1.1 (Picard's great theorem). *Let $a \in \mathbb{C}$, and let $f \in \mathrm{Hol}(\{0 < |z-a| < \delta\})$ have an essential singularity at a. There exists $w \in \mathbb{C}$ be such that the range $f(\{0 < |z-a| < r\})$ contains $\mathbb{C} \setminus \{w\}$ for all $0 < r \leq \delta$.*

Proof. We may assume that $a = 0$. Assume that there exists some $\varepsilon > 0$ such that $f \in \mathrm{Hol}(0 < |z| < \varepsilon)$ and $f(0 < |z| < \varepsilon)$ omits 2 distinct values $a, b \in \mathbb{C}$. Let $f_n(z) = f(z/n) \in \mathrm{Hol}(0 < |z| < \varepsilon)$, so $a, b \notin \mathrm{Ran}(f_n)$ for all $n \geq 1$. Apply the Montel-Caratheodory theorem to (f_n) to get a subsequence (f_{n_ν}) such that either (f_{n_ν}) converges locally uniformly in $\mathrm{Hol}(0 < |z| < \varepsilon)$ or $f_n \to \infty$ locally uniformly.

Case 1: Assume that (f_{n_ν}) converges locally uniformly in $\mathrm{Hol}(0 < |z| < \varepsilon)$. Let $K = \{z : |z| = \varepsilon/2\}$. Then $|f_{n_\nu}(z)| \leq C$ for all $z \in K$, $\nu = 1, 2, \dots$. In other words, $|f(z)| \leq C$ for $|z| = \varepsilon/(2n_\nu) \to 0$. By the maximum principle, f is bounded in a punctured neighborhood of 0, so 0 is a removable singularity for f. This is a contradiction.

Case 2: Assume that $f_{n_\nu} \to \infty$ locally uniformly. Let $g_n(z) = 1/(f_n(z) - a)$. Then g_{n_ν} is a sequence of holomorphic functions with $g_{n_\nu} \to 0$ locally uniformly. Arguing as in Case 1, we get: $g(z) = 1/(f(z) - a)$ has a removable singularity at 0 with $g(0) = 0$. So $f = a + 1/g(z)$ has a pole at 0, which is impossible. $\square$

1.2 Boundary values of harmonic functions in the disc

Theorem 1.2 (Fatou). *Let u be harmonic in D and bounded. Then the radial limits $\lim_{r \to 1^-} u(rz)$ exist for a.e. $z \in \partial D$ (with respect to 1-dimensional) Lebesgue measure on the circle. If $u = f \in \mathrm{Hol}(D)$ and $f(z) = \lim_{r \to 1^-} f(rz)$ vanishes on a set of positive measure (on the circle), then $f \equiv 0$.*

Proof. We may assume that u is real-valued. When $0 \leq r < 1$, let $\mu_r : L^1(\partial D) \to \mathbb{C}$ be the linear, continuous functional given by

$$\mu_r(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\varphi}) f(e^{i\varphi})\, d\varphi.$$

We have $|\mu_r(f)| \leq M\|f\|_{L^1}$. Then

$$\|\mu_r\|_{(L^1)^*} = \sup_{0\neq f\in L^1} \frac{|\mu_r(f)|}{\|f\|_{L^1}} \leq M, \qquad 0 \leq r < 1.$$

We can apply the Banach-Alaoglu theorem[1]: let B be a separable Banach space, and let (Λ_α) be a sequence of linear, continuous functionals $B \to \mathbb{C}$ such that $\|\Lambda_\alpha\|_{B^*} \leq C$ for all α. Then there exists a subsequence (Λ_{α_j}) such that for all $u \in B$, $(\Lambda_{\alpha_j}(u))$ converges in $\mathbb{C}$. In our case, $B = L^1$, so there exists a sequence $r_k \to 1$ such that for every $f \in L^1$, $\lim_{r_k\to 1} \mu_{r_k}(f)$ exists. Define $\mu(f)$ as this limit. We have $\mu : L^1 \to \mathbb{C}$ is linear, and $\|\mu\|_{(L^1)^*} \leq M$. Thus, $\mu \in (L^1)^*$, the space of linear, continuous functionals on L^1. This space is $L^\infty(D)$; that is, there is a $g \in L^\infty(D)$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})g(e^{i\varphi})\,d\varphi.$$

We get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e)u(r_k e^{i\varphi})f(e^{i\varphi}) \xrightarrow{k\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\varphi})f(e^{i\varphi})\,d\varphi.$$

Now $z \mapsto u(r_k z)$ is harmonic in a neighborhood of $|z| \leq 1$, so

$$u(r_k z) = \int P(z, e^{i\varphi})u(r_k e^{i\varphi})\,d\varphi \qquad \forall k, |z| < 1$$

Let $k \to \infty$. $P(z, e^{i\varphi}) \in L^1(\partial D)$, so

$$u(z) = \int_{-\pi}^{\pi} P(z, e^{i\varphi})g(e^{i\varphi})\,d\varphi.$$

In other words, u is harmonic and bounded iff u equals the Poisson integral of g for some $g \in L^\infty$. Next, we will show that $\lim_{r\to 1} u(rz) = g(z)$ for a.e. z. $\square$

We will finish the proof next time.

[1] The idea of the proof is to let take a countable dense subset (u_ν) of B and use diagonalization to find (Λ_{α_j}) such that $\lim_{j\to\infty} \Lambda_{\alpha_j}(u_\nu)$. Then extend to any $u \in B$ using $\|\Lambda_\alpha\|_{B^*} \leq C$.